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LETTER TO THE EDITOR

On algebraic structures in the non-linear principal chiral model†

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Abstract. It is demonstrated that the sum of infinitely conserved currents in the non-linear principal chiral model satisfies the Yang–Baxter relations and that the usual inverse scattering monodromy form is the zeroth power in the λ^{-n} expansion of our theory.

In the work [1], based on the correspondence principle of classical variation and the commutation relations in quantum mechanics, the commutation relations of conserved currents for the chiral model were found. Recently, intensive studies have been made on two-dimensional integrable systems. In particular, de Vega and co-workers [2, 3] point out that the commutation relations (Kac–Moody algebras) of conserved currents are in essence Yang–Baxter algebras of inverse scattering theory, and are equivalent to the monodromy relations consisting of scattering data. If we compare this result with the result obtained by Faddeev [4] and take note of the difference between the virtual vacuum and the ground state, then the classical Yang–Baxter algebras are sufficient to preserve ultralocality. Therefore, there is a connection between loop algebras, Kac–Moody algebras and real physics. In other words, it gives a physical realisation of this mathematics.

As we showed in [5], the H-transformation (HT) can be introduced to realise loop algebras in the two-dimensional model, and in [6, 7] the parametrised infinitely conserved currents were the Noether currents. In this letter, we use HT together with the correspondence principle to realise this algebraic structure. It is easy to get the results of de Vega [2, 3]. Furthermore, we are able to connect our previous discussions on algebraic structures with the Yang–Baxter algebras. Therefore it may be associated with physical inverse scattering theory. We will see that the sum of infinitely conserved currents satisfies the Yang–Baxter relations; its zeroth power of λ^{-n} is the result of [3].

As is well known, the Lagrangian density for the two-dimensional non-linear principal chiral model is

$$\mathscr{L}=\tfrac{1}{2}\operatorname{tr}(\partial\mu g^{-1}(x)\partial^{\mu}g(x)) \tag{1}$$

where $\mu=0,1$; $g(x)=g^{a}(x)I_{a}$ are functions on R^{2} with values in a Lie group with generators I_{a}; $[I_{a},I_{b}]=C^{c}_{ab}I_{c}$; $x=\{x^{0},x^{1}\}$. When the chiral field $g(x)$ makes the infinitesimal transformation, the following correspondence principle holds [1]:

$$\delta_{a}g(x)=I_{a}g(x)=\{Q_{a},g(x)\} \tag{2}$$

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where Q_a is the Noether conserved charge corresponding to the variation $\delta_a g$. Here $\{\,,\,\}$ denotes the Poisson bracket in the quantum or classical functional forms (they are the same for the chiral model), and

$$Q_a = \int j_a^0(x, t)\,\mathrm{d}x \qquad \partial\mu j_a^\mu(x, t) = 0. \tag{3}$$

Taking the variation of (2) again we get the classical variative commutator in terms of

$$[\delta_a, \delta_b]g(x) = -[I_a, I_b]g(x) = -C_{ab}^c\delta_c g(x). \tag{4}$$

Substituting (2) into (4) and noting the Jacobi identity we obtain

$$\{Q_a, Q_b\} = C_{ab}^c Q_c + (\delta_a Q_b - \delta_b Q_a). \tag{5}$$

Using (2) and (3) we can prove

$$\delta_a Q_b = \{Q_a, Q_b\}. \tag{6}$$

Thus (5) transforms into

$$\{Q_a, Q_b\} = -C_{ab}^c Q_c \tag{7}$$

or from $j_a^0(x, t) = A_a^0(x, t)$ we have

$$\{A_a^0(x, t), A_b^0(y, t)\} = -C_{ab}^c A_c^0(x, t)\delta(x-y) \tag{8}$$

which is just the result obtained in [1, 3]. Obviously, the results in the formulae (7) and (8) are unchanged under right transformation $\delta_a g(x) = -g(x)I_a$. Next, we will generalise this treatment to the local case.

We introduce the parametrised HT in terms of

$$\tilde{\delta}_a(\lambda)g(x) = -g(x)\Lambda_a(x, \lambda) \tag{9}$$

$$\Lambda_a(x, \lambda) = W(x, \lambda)I_a W^{-1}(x, \lambda) \tag{10}$$

where λ is any parameter, and W satisfies the Lax pair

$$\partial_\xi W(\xi, \eta; \lambda) = -(1+\lambda)^{-1}A_\xi(\xi, \eta)W(\xi, \eta; \lambda) \tag{11}$$

$$\partial_\eta W(\xi, \eta; \lambda) = -(1-\lambda)^{-1}A_\eta(\xi, \eta)W(\xi, \eta; \lambda) \tag{12}$$

$$\xi = x^0 + x^1 \qquad \eta = x^0 - x^1 \tag{13}$$

$$A_\xi = A_0 + A_1 \qquad A_\eta = A_0 - A_1. \tag{14}$$

The following variative commutator was given in the previous calculation [8]

$$[\tilde{\delta}_a(\lambda), \tilde{\delta}_b(\mu)]g(x) = -C_{ab}^c \frac{\mu\tilde{\delta}_c(\lambda) - \lambda\tilde{\delta}_c(\mu)}{\lambda - \mu} g(x). \tag{15}$$

Putting

$$\tilde{\delta}_a(\lambda) = \lambda\hat{\delta}_a(\lambda) \qquad \tilde{\delta}_a(\mu) = \mu\hat{\delta}_b(\mu) \tag{16}$$

we have

$$[\hat{\delta}_a(\lambda), \hat{\delta}_b(\mu)]g(x) = C_{ab}^c \frac{\hat{\delta}_c(\lambda) - \hat{\delta}_c(\mu)}{\lambda - \mu} g(x). \tag{17}$$

As shown in [6, 7], the Noether-type currents exist under HT. Their time component is a parametrised conserved charge

$$Q_a(\lambda)=\operatorname{tr}(I_aQ(\lambda)) \tag{18}$$

$$Q(\lambda)=\sum_{n=0}^{\infty}\lambda^{-n}Q^{(n)}=\sum_{n=0}^{\infty}\lambda^{-n}\int j_0^{(n)}(x,t)\,\mathrm{d}x \tag{19}$$

and the first few expressions for $j_\mu^{(n)}$ were

$$\begin{aligned}
&j_\mu^{(0)}=A_\mu=g^{-1}\partial\mu g \qquad j_\mu^{(1)}=[A_\mu,\chi^{(1)}]+\varepsilon_{\mu\nu}A^\nu-\tfrac{1}{2}\varepsilon_{\mu\nu}[\partial^\nu\chi^{(1)},\chi^{(1)}]\\
&j_\mu^{(2)}=[A_\mu,\chi^{(2)}]+\tfrac{1}{2}[[A_\mu,\chi^{(1)}],\chi^{(1)}]-\varepsilon_{\mu\nu}[\partial^\nu\chi^{(1)},\chi^{(2)}]\\
&\qquad\quad-\tfrac{1}{6}\varepsilon_{\mu\nu}[[\partial^\nu\chi^{(1)},\chi^{(1)}],\chi^{(1)}]+\varepsilon_{\mu\nu}[A^\nu,\chi^{(1)}]
\end{aligned} \tag{20}$$

where $\chi^{(1)}$, $\chi^{(2)},\ldots,$ are operator functions obtained by expanding $W(\lambda)$ in terms of λ^{-1}.

Analogously to (2), the following correspondence principle for classical variation is established under HT:

$$\tilde{\delta}_a(\lambda)g(x)=-g(x)\Lambda_a(x,\lambda)=\{Q_a(\lambda),g(x)\}. \tag{21}$$

Based on this equation and the Jacobi identity, we derive

$$\begin{aligned}
&[\tilde{\delta}_a(\lambda),\tilde{\delta}_b(\mu)]g(x)\\
&\quad=-\{\{Q_a(\lambda),Q_b(\mu)\},g(x)\}+\{(\tilde{\delta}_a(\lambda)Q_b(\mu)-\tilde{\delta}_b(\mu)Q_a(\lambda)),g(x)\}.
\end{aligned} \tag{22}$$

This is the generalisation of [1]. Next, we will find $\tilde{\delta}_a(\lambda)Q(\mu)$. From $A_0(x)=g^{-1}(x)\partial_0 g(x)$, equation (21) and identity

$$g^{-1}\{Q_a(\lambda),\partial_0 g\}-g^{-1}\{Q_a(\lambda),g\}g^{-1}\partial_0 g=\{Q_a(\lambda),g^{-1}\partial_0 g\}$$

it is easy to prove that

$$\tilde{\delta}_a(\lambda)A_0(x,t)=\{Q_a(\lambda),A_0(x,t)\} \tag{23}$$

as well as

$$\tilde{\delta}_a(\lambda)L_\xi(\mu)=\{Q_a(\lambda),L_\xi(\mu)\} \tag{24}$$

$$\tilde{\delta}_a(\lambda)L_\eta(\mu)=\{Q_a(\lambda),L_\eta(\mu). \tag{25}$$

From one of the Lax pair (11) and equation (24) we have

$$\partial_\xi\tilde{\delta}_a(\lambda)W(\mu)=\{Q_a(\lambda),L_\xi(\mu)\}W(\mu)+L_\xi(\mu)(\tilde{\delta}_a(\lambda)W(\mu)) \tag{26}$$

whose formal solution is

$$\tilde{\delta}_a(\lambda)W(\mu)=\{Q_a(\lambda),W(\mu)\}. \tag{27}$$

Expanding W in λ^{-1}, $W(x,t;\lambda)=\sum_{n=0}^{\infty}\lambda^{-n}W^{(n)}(x,t)$. Equation (27) then means that

$$\tilde{\delta}_a(\lambda)W^{(n)}=\{Q_a(\lambda),W^{(n)}\}. \tag{28}$$

Starting frcm this formula, taking account of $\tilde{\delta}_a(\lambda)(F_1F_2\ldots)=\{Q_a(\lambda),F_1F_2\ldots)$, $\delta_a(\lambda)[F_i,F_j]=\{Q_a(\lambda),[F_i,F_j]\}$ and using (19) and (20) we can prove

$$\tilde{\delta}_a(\lambda)Q_b(\mu)=\{Q_a(\lambda),Q_b(\mu)\}. \tag{29}$$

Substituting (29) into (22) and introducing $Q_a(\lambda)=\lambda\hat{Q}_a(\lambda)$, we get

$$\{\hat{Q}_a(\lambda),\hat{Q}_b(\mu)\}=C_{ab}^{c}\frac{\hat{Q}_c(\lambda)-\hat{Q}_c(\mu)}{\lambda-\mu}. \tag{30}$$

This is a general Poisson bracket for all powers of λ^{-1} for infinitely conserved charges.

As indicated by Reshetikhin and Faddeev [9], if any matrices A_a and r exist and satisfy

$$[r, A_a \otimes 1 + 1 \otimes A_a] = 0 \tag{31}$$

$$[r, A^c \otimes 1] = C^c_{ab} A^a \otimes A^b \tag{32}$$

then we may write (30) in the following direct-product form after standard calculations:

$$\{\hat{Q}(\lambda) \otimes \hat{Q}(\mu)\} = \left[\frac{r}{\lambda - \mu}, \hat{Q}(\lambda) \otimes 1 + 1 \otimes \hat{Q}(\mu)\right]. \tag{33}$$

Even though solutions satisfying (31) and (32) are few, we can choose a pole-type solution, such as

$$A_a = K_{ab} I^b \qquad r = K_{ab} I^a \otimes I^b \tag{34}$$

where K_{ab} is a Killing metric. It is shown that the sum of $j_0^{(n)}(x, t)$ satisfies a simple current algebraic relation, the Yang–Baxter relation, although the expressions for $j_\mu^{(n)}$ in formula (20) are very complicated. This result is more general than those of [2, 3]. In fact, if we take the zeroth power of λ^{-1} in $j(x, t; \lambda)$, then we obtain the result of [2, 3]. In addition, we are also able to get the well known relation

$$\{\mathrm{tr}\, T(\lambda), \mathrm{tr}\, T(\mu)\} = 0 \tag{35}$$

where $T(\lambda)$ is usually called the monodromy matrix obtained from $W(x, t; \lambda)|_{x\to\infty} = T(\lambda)$ which is consistent with $A_\mu(x, t)|_{|x|\to\infty} = 0$.

Our discussion extends the ideas of [1–3] to the two-dimensional Lax pair system. The results obtained here are general Yang–Baxter algebras. Its zeroth power of λ^{-1} in the conserved current corresponds to (8), which can be transformed into the monodromy form of the inverse scattering theory. Other powers of these relations are not simple. They only reflect unknown current algebras. Since basic commutation relations hold for $A_a^0(x, t)$, the Poisson brackets are consistent with the commutation relations in quantum mechanics (differing by i). Its difference will appear upon introducing the Schwinger term. If we extend the discussion of this paper to include the Schwinger term, the Witten anomaly and the Riemann–Hilbert transformation, then we can expect more interesting results.

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